

Spacetime Symmetries and Kepler's Third Law

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Abstract

In general relativity, the generators of symmetries are called Killing vector fields. The spacetime geometry of a system of two point masses moving on a circular orbit has a helical Killing vector (HKV). We show how Kepler's third law for circular orbits, and its generalization in post-Newtonian theory, can be recovered from a simple, covariant condition on the norm of that HKV. This unusual derivation can be used to illustrate some concepts of prime importance in a general relativity course, including those of Killing vector, covariance, coordinate dependence, and gravitational redshift.

I. INTRODUCTION AND SUMMARY

One could hardly overstate the role played by symmetries in physics. Symmetry considerations can very often drastically simplify the process of solving a given physics problem. At a more fundamental level, symmetries are deeply connected to the existence of conserved quantities (*via* Noether's theorem), and they considerably restrain the span of admissible field theories in modern theoretical physics. Within Einstein's theory of general relativity (GR), which describes gravity as a manifestation of the curvature of spacetime, the generators of symmetries are called Killing vectors. They are widely used in current research in gravitation theory, and are an essential part of one's education in GR.^{1–6}

Gravitational radiation is one of the most important predictions of GR. Observing gravitational waves would have a tremendous impact on physics, astrophysics and cosmology.⁷ A worldwide effort is currently underway to achieve the first direct detection, using kilometer-scale ground-based interferometers such as LIGO,⁸ Virgo,⁹ GEO600,¹⁰ and LCGT,¹¹ as well as future space-based antennas, such as the planned ESA-led mission eLISA/NGO.¹² Binary systems composed of compact objects (neutron stars or black holes) are among the most promising sources of gravitational radiation. However, the detection and analysis of these exceedingly weak signals require very accurate theoretical predictions, for use as template waveforms to be compared to the output of the detectors.¹³ Hence, modeling the orbital dynamics and gravitational-wave emission of compact binary systems is a timely problem in relativistic astrophysics.

Except for the occurrence of a gradual inspiral driven by gravitational radiation-reaction, the orbits of stellar-mass compact binaries can be considered to be circular, to a very high degree of approximation.¹⁴ Mathematically, the approximation of an exactly closed circular orbit translates into the existence of a *helical Killing vector* K^α , along the orbits of which the spacetime geometry is invariant. Given the high astrophysical relevance of this approximation, helically symmetric spacetimes have been studied extensively.^{15–29}

In this paper, we shall consider two non-spinning compact objects moving on exactly circular orbits. These will be modeled as point masses m_A (with $A = 1, 2$), a prescription commonly adopted in the field of gravitational-wave source modeling.^{30,31} We will prove that the gradient of the norm K^2 of the helical Killing vector K^α must vanish along the worldlines of the particles:

$$(\nabla_\alpha K^2)_A = 0. \quad (1.1)$$

We will then show how this simple, geometric result can be used to derive the main relation encoding the orbital dynamics of the binary, namely Kepler's third law (for circular orbits), and its generalization in post-Newtonian theory. This unusual derivation can be used to illustrate numerous concepts of prime importance in a GR course, including those of Killing vector, covariance, coordinate dependence, and gravitational redshift.

This paper is organized as follows: In Sec. II we summarize some well-known, yet useful properties of Killing vectors. We then discuss, in Sec. III, the physics of binary systems of point masses moving along circular orbits, introducing the notion of redshift observable in Sec. III A, and proving the relation (1.1) in Sec. III B. The derivation of Kepler's third law, and its generalization in post-Newtonian theory, are discussed in Secs. IV A and IV B, respectively. Finally, Sec. V is devoted to some further comments of physical relevance on the norm of the HKV and its link to the redshift observable.

II. SOME PROPERTIES OF KILLING VECTORS

For the convenience of the reader, we start by summarizing a few well-known, elementary properties of Killing vectors, which will be used extensively throughout this paper. The main property of a Killing vector field k^α is that it satisfies Killing's equation

$$\mathcal{L}_k g_{\alpha\beta} = \nabla_\alpha k_\beta + \nabla_\beta k_\alpha = 0, \quad (2.1)$$

where ∇_α is the covariant derivative compatible with the spacetime metric $g_{\alpha\beta}$, and \mathcal{L}_k is the Lie derivative along k^α . Equation (2.1) expresses the invariance of the spacetime geometry along the integral curves of the Killing vector.

In addition, Killing vectors straightforwardly provide well-defined conserved quantities: Let u^α be the four-velocity of a test particle, tangent to its worldline, and normalized such that $g_{\alpha\beta} u^\alpha u^\beta = -1$. Then the scalar product $s \equiv k^\alpha u_\alpha$ is a constant of the motion along the timelike geodesic followed by the test mass:

$$\dot{s} \equiv u^\beta \nabla_\beta (u^\alpha k_\alpha) = \dot{u}^\alpha k_\alpha + u^\alpha u^\beta \nabla_\beta k_\alpha = 0. \quad (2.2)$$

This immediately follows from the geodesic equations of motion, $\dot{u}^\alpha = 0$, and the antisymmetry property of the tensor $\nabla_\alpha k_\beta$. Two familiar examples of such conserved quantities in curved spacetime are the energy per unit mass $e = -k_{(t)}^\alpha u_\alpha$ and angular momentum per unit mass $j = k_{(\varphi)}^\alpha u_\alpha$ of a test particle in orbit around a rotating black hole, where $k_{(t)}^\alpha$ and $k_{(\varphi)}^\alpha$ are Killing vectors associated with the stationarity and axisymmetry of the Kerr metric.

III. HELICALLY SYMMETRIC POINT-PARTICLE SPACETIMES

We now consider a binary system of non-spinning compact objects moving on a circular orbit. The neutron stars or black holes will be modeled as point particles with constant masses m_A (with $A = 1, 2$), and four-velocities u_A^α normalized to $g_{\alpha\beta}^A u_A^\alpha u_A^\beta = -1$. Note that the point masses m_A are not test particles; their stress-energy tensor curves the geometry *via* Einstein's field equations. Their motion obeys the standard geodesic equations, $\dot{u}_A^\alpha = 0$, albeit in a *regularized* metric $g_{\alpha\beta}^A$ such that the divergent self-fields of the point particles have been carefully subtracted.^{32–35}

The spacetime geometry of that binary system is neither stationary, nor axisymmetric; however it is invariant along the orbits of a helical Killing vector (HKV) K^α . Far away from the binary system, this field has the asymptotic behavior

$$K^\alpha \rightarrow (\partial_t)^\alpha + \Omega (\partial_\varphi)^\alpha, \quad (3.1)$$

where the four-vectors $(\partial_t)^\alpha$ and $(\partial_\varphi)^\alpha$ are part of the coordinate basis of an inertial frame of reference. The constant Ω is interpreted as the circular-orbit frequency of the binary system. Heuristically, K^α can be seen as the generator of time translations in a co-rotating frame. In particular, if you imagine yourself “sitting” on one of the particles, orbiting around the companion star, then you would observe no change in the local geometry. In other words, the metric is invariant along the worldlines of the particles: $\mathcal{L}_{u_A} g_{\alpha\beta} = 0$. This implies that the four-velocities of the particles must be aligned with the HKV evaluated at their respective coordinate locations:

$$u_A^\alpha = u_A^T K_A^\alpha \iff K_A^\alpha = z_A u_A^\alpha, \quad (3.2)$$

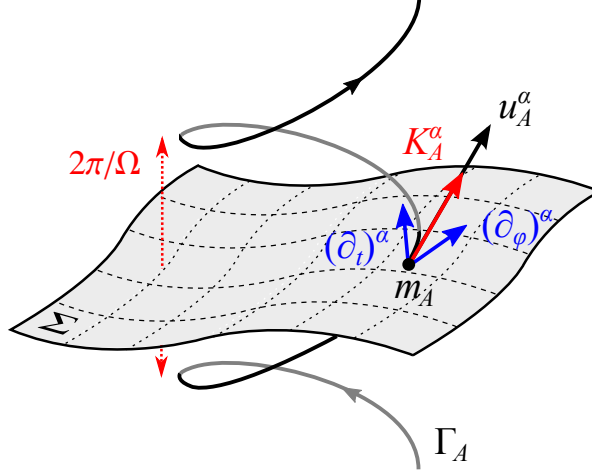


FIG. 1. Spacetime diagram picturing a binary system of point masses m_A ($A = 1, 2$) on a circular orbit with constant azimuthal frequency Ω . The helical Killing vector $K^\alpha = (\partial_t)^\alpha + \Omega(\partial_\phi)^\alpha$ is aligned with the four-velocities u_A^α tangent to the worldlines Γ_A of the particles. The redshift observables are given by the scalar products $z_A = -K_A^\alpha u_{\alpha}^A$.

where we pose $z_A \equiv 1/u_A^T$. See Fig. 1 for an illustration. As a coefficient of proportionality between two four-vectors, z_A must be a scalar. It can be assigned several physical interpretations, which, in a GR course, would provide the opportunity to discuss the important notion of coordinate invariance.

A. Redshift observable

First, contracting the relation (3.2) with the four-velocity u_A^α , and remembering Eq. (2.2), we notice that z_A is a constant of the motion associated with the helical symmetry:

$$z_A = -K_A^\alpha u_{\alpha}^A = \text{const.} \quad (3.3)$$

Furthermore, it can easily be established that z_A measures the redshift of lights rays emitted from particle A , and received far away from the binary, along the helical symmetry axis perpendicular to the orbital plane:²⁶ Let p^α be a four-vector tangent to the worldline of such a light ray, *e.g.* the four-momentum of the associated “photon”. Then the ratio of the photon energy at reception and emission is given by

$$\frac{\mathcal{E}_{\text{rec}}}{\mathcal{E}_{\text{em}}} = \frac{(u^\alpha p_\alpha)_{\text{rec}}}{(u^\alpha p_\alpha)_{\text{em}}} = \frac{(K^\alpha p_\alpha)_{\text{rec}}}{u_A^T (K^\alpha p_\alpha)_{\text{em}}} = z_A. \quad (3.4)$$

We made use of the equality $u_{\text{rec}}^\alpha = K_{\text{rec}}^\alpha = (\partial_t)^\alpha$ between the four-velocity of the observer and the HKV, of the relationship (3.2) at the location of the emitter, and of the conservation of $K^\alpha p_\alpha$ along the null geodesic of the photon. (The proof of that last point is identical to that given in Eq. (2.2), with the substitutions $u^\alpha \rightarrow p^\alpha$ and $k^\alpha \rightarrow K^\alpha$.) See Fig. 2 for an illustration of this Gedankenexperiment. Following Detweiler, we shall thus refer to z_A as the “redshift observable”.²⁶

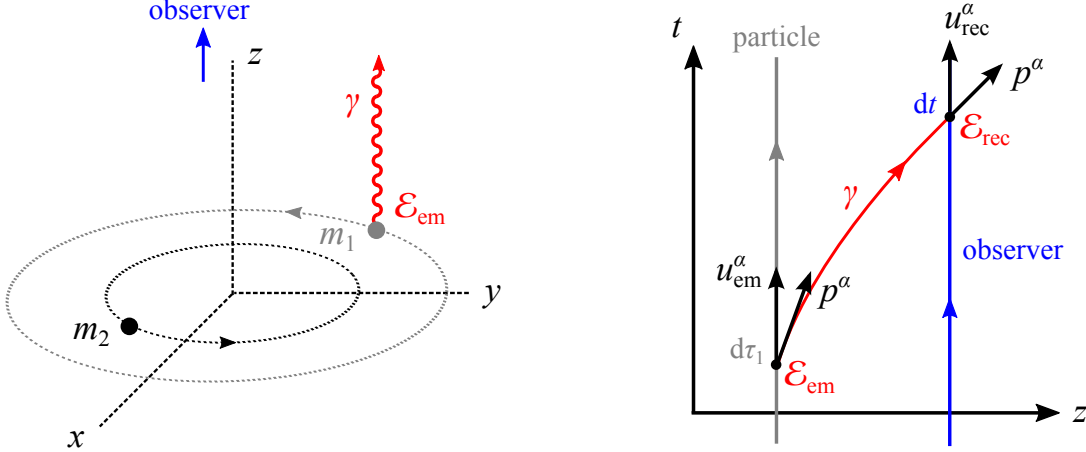


FIG. 2. A photon γ with four-momentum p^α is emitted from the particle m_1 with four-velocity $u_{\text{em}}^\alpha = u_1^\alpha$, and received far away from the binary system, by a distant inertial observer with four-velocity $u_{\text{rec}}^\alpha = (\partial_t)^\alpha$, along the helical symmetry axis z perpendicular to the orbital plane. Detweiler's observable z_1 measures the redshift $\mathcal{E}_{\text{rec}}/\mathcal{E}_{\text{em}}$ of the photon, or equivalently the ratio $d\tau_1/dt$.

Finally, in a cylindrical coordinate system $\{ct, \rho, \varphi, z\}$ adapted to the helical symmetry, *i.e.* such that the expression $K^\alpha = (\partial_t)^\alpha + \Omega (\partial_\varphi)^\alpha$ holds everywhere,³⁶ and not merely far away from the binary, Eq. (3.2) implies

$$z_A = (u_A^0)^{-1} = \frac{d\tau_A}{dt}. \quad (3.5)$$

Hence the redshift observable coincides with the inverse time component of the four-velocity of the particle, or equivalently with the ratio of the proper times elapsed along the worldlines of the particle and of the distant inertial observer (*cf.* Fig. 2). This last interpretation is in agreement with the usual notion of redshift (gravitational redshift and/or Doppler effect).

B. Geometric characterization of the binary dynamics

We now have at our disposal all of the concepts and results necessary to prove Eq. (1.1), namely that the spacetime gradient of the norm $K^2 \equiv g_{\alpha\beta} K^\alpha K^\beta$ of the HKV must vanish at the location of each particle. The derivation goes as follows:³⁷

$$\frac{1}{2}(\nabla_\alpha K^2)_A = K_A^\beta (\nabla_\alpha K_\beta)_A = -z_A u_A^\beta (\nabla_\beta K_\alpha)_A = -z_A (\dot{z}_A u_\alpha^A + z_A \dot{u}_\alpha^A) = 0. \quad (3.6)$$

We successively made use of the relationship (3.2) between K_A^α and u_A^α , of Killing's equation, of the geodesic equations of motion, and of the fact that z_A is a constant of the motion. Although pretty straightforward, that proof would provide a good exercise for students.

The formula (1.1) is quite simple and elegant; it is covariant, and only makes reference to well-defined geometrical concepts in GR. It implies, in particular, that the function $K^2(\mathbf{x})$ has extrema at the coordinate locations of the particles (see Fig. 3). Furthermore, we are going to show that (1.1) encodes a well-known result of classical mechanics, namely Kepler's third law (for circular orbits), and its generalization in post-Newtonian theory.

IV. KEPLER'S THIRD LAW AND ITS RELATIVISTIC GENERALIZATION

A. Newtonian gravity

For simplicity, we start by restricting ourselves to the Newtonian approximation of the full theory of GR. This corresponds to the leading-order results in the formal limit $c^{-1} \rightarrow 0$. In that weak-field, non-relativistic approximation, the spacetime metric expressed in cylindrical coordinates $\{ct, \rho, \varphi, z\}$ takes the form

$$ds^2 = \left(-1 + \frac{2U}{c^2}\right) c^2 dt^2 + d\rho^2 + \rho^2 d\varphi^2 + dz^2, \quad (4.1)$$

where the Newtonian gravitational potential reads $U(t, \mathbf{x}) = \sum_A Gm_A/|\mathbf{x} - \mathbf{y}_A(t)|$, with $\mathbf{y}_A(t)$ the coordinate trajectory of the mass m_A . In our convenient coordinate system such that $K^\alpha = (\partial_t)^\alpha + \Omega (\partial_\varphi)^\alpha$, the norm of the HKV reads $K^2 = g_{00} + 2\Omega g_{0\varphi}/c + \Omega^2 g_{\varphi\varphi}/c^2$. With the explicit expression (4.1) of the metric, this gives

$$K^2 = -1 + \frac{2U}{c^2} + \frac{\rho^2 \Omega^2}{c^2}. \quad (4.2)$$

In order to apply the result (1.1), we need first to compute the partial derivatives of the scalar K^2 . Using the Cartesian coordinates $\{x^i\}$ associated with the cylindrical coordinates $\{\rho, \varphi, z\}$ in the usual way, namely $x^1 = \rho \cos \varphi$, $x^2 = \rho \sin \varphi$, and $x^3 = z$, we have

$$\partial_t K^2 = \frac{2}{c^2} \partial_t U \quad \text{and} \quad \partial_i K^2 = \frac{2}{c^2} (\partial_i U + \rho \Omega^2 n^i), \quad (4.3)$$

where $\mathbf{n} = (\cos \varphi, \sin \varphi, 0)$ is the unit vector in the orbital plane $z = 0$. Focusing first on the spatial components, the relationship (1.1) implies

$$-\rho_A \Omega^2 n_A^i = (\partial_i U)_A. \quad (4.4)$$

We recognize Newton's third law expressing the equality of the centripetal acceleration of body A and of the Newtonian gravitational force exerted on the body A by the body $B \neq A$. Computing the forces explicitly, we find³⁸

$$\rho_1 \Omega^2 = \frac{Gm_2}{r^2} \quad \text{and} \quad \rho_2 \Omega^2 = \frac{Gm_1}{r^2}, \quad (4.5)$$

where $r \equiv |\mathbf{y}_1 - \mathbf{y}_2| = \rho_1 + \rho_2$ is the coordinate separation between the two point masses. We may then add up Eqs. (4.5), or remember that in the center-of-mass frame $\rho_1 = r m_2/m$ and $\rho_2 = r m_1/m$, with $m = m_1 + m_2$ the total mass of the binary. Both solutions yield Kepler's third law

$$\Omega^2 = \frac{Gm}{r^3}, \quad (4.6)$$

which is recovered here in the particular case of circular motion. (Our derivation cannot be extended to generic eccentric orbits, for which the helical symmetry is lost.) On the other hand, the time component of Eq. (1.1) is identically satisfied, as $(\partial_t U)_A \propto \mathbf{v}_B \cdot \mathbf{n}_{12}$ vanishes for circular orbits, $\mathbf{v}_B = d\mathbf{y}_B/dt$ being the coordinate velocity of the particle $B \neq A$, and \mathbf{n}_{12} the unit vector pointing from m_2 to m_1 .

B. Post-Newtonian gravity

Equation (1.1) being valid beyond the Newtonian limit, the result (4.6) can be extended to include corrections coming from the full theory, by keeping known post-Newtonian terms in the metric $g_{\alpha\beta}$. For example, at the first post-Newtonian (1PN) order, *i.e.* including the general relativistic corrections $\mathcal{O}(c^{-2})$ to the Newtonian expression (4.1), the metric reads (in Cartesian-like harmonic coordinates)³⁹

$$g_{00} = -1 + \frac{2Gm_1}{c^2 r_1} + \frac{1}{c^4} \left[\frac{Gm_1}{r_1} (4\mathbf{v}_1 \cdot \mathbf{v}_1 - (\mathbf{n}_1 \cdot \mathbf{v}_1)^2) - \frac{2G^2 m_1^2}{r_1^2} + G^2 m_1 m_2 \left(-\frac{2}{r_1 r_2} - \frac{r_1}{2r_{12}^3} + \frac{r_1^2}{2r_2 r_{12}^3} - \frac{5}{2r_2 r_{12}} \right) \right] + (1 \leftrightarrow 2) + \mathcal{O}(c^{-6}), \quad (4.7a)$$

$$g_{0i} = -\frac{4Gm_1}{c^3 r_1} v_1^i - \frac{4Gm_2}{c^3 r_2} v_2^i + \mathcal{O}(c^{-5}), \quad (4.7b)$$

$$g_{ij} = \delta^{ij} \left(1 + \frac{2Gm_1}{c^2 r_1} + \frac{2Gm_2}{c^2 r_2} \right) + \mathcal{O}(c^{-4}), \quad (4.7c)$$

where δ^{ij} is the usual Kronecker symbol, $r_{12} = |\mathbf{y}_1 - \mathbf{y}_2| = r$ is the coordinate separation, $r_A = |\mathbf{x} - \mathbf{y}_A|$, and $\mathbf{n}_A = (\mathbf{x} - \mathbf{y}_A)/r_A$. Transforming that metric to cylindrical coordinates, and repeating the calculation detailed in Sec. IV A, we recover from $(\partial_i K^2)_A = 0$ the known generalization of Kepler's third law at 1PN order (in harmonic coordinates), namely³⁹

$$\Omega^2 = \frac{Gm}{r^3} \left\{ 1 + (-3 + \nu) \frac{Gm}{c^2 r} + \mathcal{O}(c^{-4}) \right\}. \quad (4.8)$$

At that order of approximation, the calculation involves a crucial contribution coming from the general relativistic frame-dragging effect, through the metric component $g_{0\varphi}$. The 1PN coefficient $(-3 + \nu)$ in Eq. (4.8) depends on the symmetric mass ratio $\nu \equiv m_1 m_2 / m^2$, such that $\nu = 1/4$ for an equal-mass binary, and $\nu \rightarrow 0$ in the extreme mass ratio limit. (The relation $(\partial_t K^2)_A = 0$ is still found to be satisfied identically.)

In a GR course, this derivation would provide the occasion to discuss the key notions of covariance and coordinate dependence using a concrete example: Being covariant, Eq. (1.1) conveys a physically meaningful result, independent of a particular choice of coordinates. By contrast, the generalized version (4.8) of Kepler's third law is coordinate dependent; the 1PN coefficient could be different from $(-3 + \nu)$ if the relationship between the invariant frequency Ω and the coordinate dependent separation r was expressed in another coordinate system. But the precise way in which the function $\Omega(r)$ changes, depending on the coordinate system used to write the post-Newtonian metric, is precisely encoded in the covariance of Eq. (1.1). Finally, we note that while the relationship between Ω and r is coordinate dependent, the functions $z_A(\Omega)$ are coordinate invariant; they have recently been computed up to very high orders in the post-Newtonian approximation.^{26–29}

V. HELICAL KILLING VECTOR AND REDSHIFT

From the Newtonian result (4.2), we have the asymptotic behavior $K^2 \sim (\rho \Omega / c)^2 > 0$ in the limit $\rho \rightarrow +\infty$, which indicates that the HKV is spacelike far away from the helical symmetry axis. Close to the binary system however, and along the worldlines of the particles

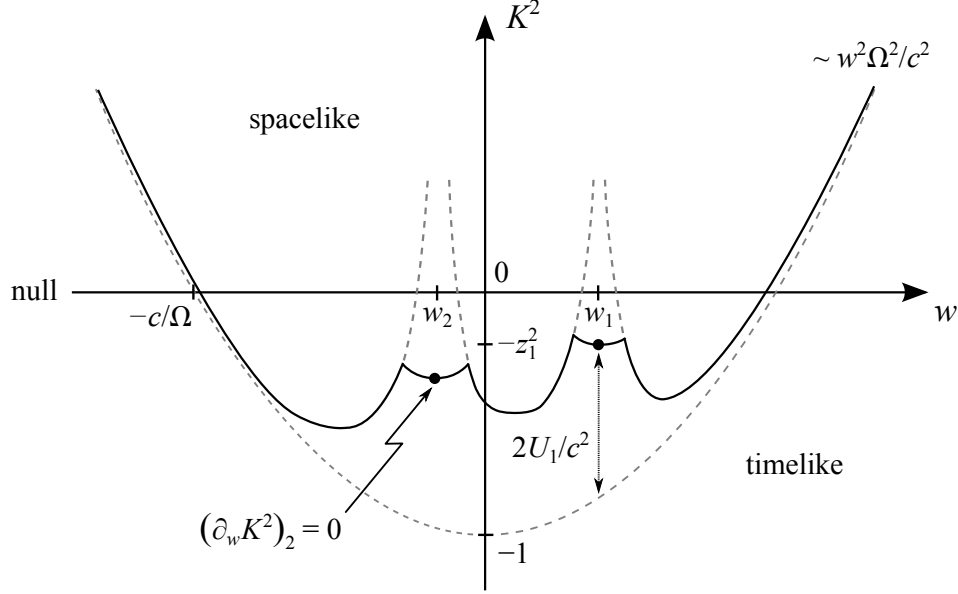


FIG. 3. The norm K^2 of the helical Killing vector K^α as a function of the coordinate w along the direction joining the two particles, within the orbital plane.

in particular, the HKV is timelike ($K^2 < 0$) [remember Eq. (3.2)]. In between, there must exist a worldtube over which the HKV is null ($K^2 = 0$). This hypersurface is usually referred to as the “light cylinder”. In the flat spacetime limit $m_A \rightarrow 0$, its radius is simply $\rho = c/\Omega$. (In the language of classical mechanics, this is the distance from the axis of rotation for which the velocity of an observer rotating at the constant angular rate Ω reaches the vacuum speed of light c .)

Furthermore, when evaluated at the locations of the particles themselves, the norm of the HKV is directly related to the redshift observables. Indeed, from Eq. (3.2) we immediately get $z_A^2 = -K_A^2$. In the Newtonian limit, we thus have

$$z_A^2 = 1 - \frac{2U_A}{c^2} - \frac{\rho_A^2 \Omega^2}{c^2}. \quad (5.1)$$

This result is consistent with the interpretation of z_A as a measure of the redshift of light rays, as discussed in Sec. III A. The observable z_A has two contributions: (i) a term proportional to the Newtonian potential U_A evaluated at the coordinate location of particle A , which gives the gravitational redshift, or Einstein effect, and (ii) a term involving the relative velocity $v_A = \rho_A \Omega$ with respect to the distant observer, yielding a transverse Doppler effect. (In the flat spacetime limit $m_A \rightarrow 0$, we recover the special relativistic result $z_A = \sqrt{1 - v_A^2/c^2}$.)

Based on the previous qualitative discussion, the function $K^2(w)$ is depicted schematically in Fig. 3, with $w = \rho \cos(\varphi - \varphi_1)$ the coordinate along the direction joining the two particles, within the orbital plane. (The divergent Newtonian self-fields of the point masses are shown in dashed lines; these are well-known artifacts of the use of point particles to model the actual physical compact stars, which are extended objects.) A more quantitative analysis of the function $K^2(\mathbf{x})$ in Newtonian (or post-Newtonian) gravity, in and out of the orbital plane, could be a useful exercise for students. In particular, they could be asked to plot that relation for different sets of values for $\{\Omega, m_1, m_2\}$.

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- ³⁶ In such a coordinate system, any scalar (or component of a tensor) F that respects the helical symmetry must satisfy $\mathcal{L}_K F = (\partial_t + \Omega \partial_\varphi) F = 0$, and thus depend on the coordinate time t and the azimuthal angle φ only through the combination $\varphi - \Omega t$.
- ³⁷ The covariant derivative $(\nabla_\beta K_\alpha)_A$ being evaluated along the four-velocity u_A^β , we can replace the HKV K_α by its value along the worldline of particle A , namely $K_\alpha^A = z_A u_\alpha^A$.
- ³⁸ The singular self-forces of the point masses ought to be subtracted. This can be done by means of a suitable regularization method, such as dimensional regularization.
- ³⁹ L. Blanchet, G. Faye, and B. Ponsot, “Gravitational field and equations of motion of compact binaries to 5/2 post-Newtonian order,” *Phys. Rev. D* **58**, 124002-1–20 (1998).